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# Logarithmic perturbation theory for quasinormal modes

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**Abstract.** Logarithmic perturbation theory (LPT) is developed and applied to quasinormal modes (QNMs) in open systems. QNMs often do not form a complete set, so LPT is especially convenient because summation over a complete set of unperturbed states is not required. Attention is paid to potentials with exponential tails, and the example of a Pöschl–Teller potential is briefly discussed. A numerical method is developed that handles the exponentially large wavefunctions which appear in dealing with QNMs.

## 1. Introduction

### 1.1. Logarithmic perturbation theory

Eigenvalue problems of the type

$$H\phi = \lambda\phi \quad (1.1)$$

occur in many branches of physics; here  $\lambda$  may be related to the frequency  $\omega$  by  $\lambda = \omega$  (Schrödinger equation), or by  $\lambda = \omega^2$  (wave equation or Klein–Gordon equation). Perturbation theory is useful for systems that depart slightly from an ideal solvable configuration. Apart from the standard Rayleigh–Schrödinger perturbation theory (RSPT), a useful alternative focuses not on  $\phi$  itself, but on its logarithmic derivative  $f = \phi'/\phi$ . Known as logarithmic perturbation theory (LPT) [1–4], this method is commonly applied to one-dimensional (1D) bound state problems, especially the ground state. For excited states, one either has to first factor out the zeros [5], or detour around them in the complex plane [6]. LPT has also been developed for bound states in three dimensions [7]. LPT avoids sums over intermediate states, and comparison with RSPT can lead to useful sum rules [8].

The bound states, or normal modes (NMs), are solutions with  $\phi \rightarrow 0$  as  $x \rightarrow \pm\infty$ . Other boundary conditions are also important physically. Scattering states and the phase shift can be handled by LPT using only on-shell information [9]. In this paper, we develop LPT for wavefunctions that are outgoing at infinity—quasinormal modes (QNMs).

### 1.2. Quasinormal modes

In conservative systems, NMs are factorized solutions  $\Phi(x, t) = e^{-i\omega t} \phi(x)$  with  $\phi$  satisfying an eigenvalue equation such as (1.1) and nodal boundary conditions at  $x \rightarrow \pm\infty$ . The counterparts in open systems are QNMs; these factorized solutions satisfy outgoing wave boundary conditions at  $x \rightarrow \pm\infty$ , so that  $\text{Im } \omega \equiv -\gamma < 0$ .

QNMs are important from many points of view. A laser is often discussed in terms of its ‘modes’, i.e. the spectral lines with finite widths  $\gamma$ , which are precisely these QNMs [10]. Quantum-mechanical resonances are likewise central to scattering [11–14], and as intermediate states in high-order transitions. Gravitational waves from the vicinity of a black hole are likely to be detected in the next decade by facilities such as LIGO and VIRGO [15]. The radial wavefunction describing the propagation of gravitational waves in any angular momentum sector satisfies a Klein–Gordon equation with a potential  $V(x)$  [16, 17]. Theoretical studies [18, 19] show that, at least for an intermediate time domain, the waves are dominated by a ringing signal, which is readily identified as the superposition of QNMs [18–20]. If the relationship between the characteristics of the ringing signal (i.e. the QNMs) and the spacetime curvature could be better understood, gravitational waves have the prospect of becoming a novel astronomical probe. In these cases, the background is a Schwarzschild metric plus perturbations (e.g. due to an accretion disc), so perturbative treatments will be useful.

In the present context, three properties of QNMs should be emphasized. First, their numerical determination is notoriously difficult. This is most simply seen in the ‘shooting’ algorithm: choose an arbitrary  $\omega$ , integrate from one end (say  $x = 0$  for a half-line problem or a full-line problem with definite parity), identify the coefficient of the ‘wrong’ solution at the other end (say  $x \rightarrow \infty$ ), and vary  $\omega$  until this coefficient is zero. For NMs, the exponentially large ‘wrong’ solution is readily identified. For QNMs, the ‘wrong’ solution, which is  $O(e^{-2\gamma x})$  relative to the ‘right’ solution, is difficult to extract, especially when  $\gamma$  is large. The numerical difficulties make perturbation methods even more relevant than would be the case for NMs.

Secondly, RSPT is inapplicable for two reasons. Its usual derivation relies on the Hermiticity of the system, which is now lost. Moreover, because the QNMs are in general not complete, one cannot sum over intermediate states. Even in circumstances where the QNMs turn out to be complete [12, 13, 21–26], a scheme such as LPT would still have definite advantages, because it makes no reference to the higher states with large  $\gamma$ .

Thirdly, it is readily shown that any QNM can have at most one node on the real  $x$ -axis. Except for the origin for the odd-parity sector of symmetric potentials, there is no reason why any root of the *complex* equation  $\phi(x) = 0$  should lie on the real  $x$ -axis; those cases that do are therefore ‘accidental’ in the sense that they occur only for specific values of the parameters defining the potential  $V(x)$ —in other words on a set of measure zero in parameter space. Therefore the nodal problem which plagues LPT for excited NMs is here generally absent.

### 1.3. Outline

Section 2 develops LPT for QNMs, and discusses the generalized norm that emerges as a result. The most explicit general form for the second-order correction, together with an illustrative example, are given in section 3, focusing on those cases where both the original potential and the perturbation have finite support. The situation becomes slightly more complicated if the potentials have tails, and the case of exponential tails is discussed in section 4. A conclusion is given in section 5.

## 2. Perturbation theory

### 2.1. Formalism for the eigenvalue

We deal with the Klein–Gordon equation:

$$[\partial_x^2 - V(x) + \omega^2]\phi(x) = 0. \quad (2.1)$$

The Schrödinger equation is included by simply relabelling  $\omega^2 \mapsto \omega$ . The logarithmic derivative  $f(x) = \phi'(x)/\phi(x)$  satisfies the Riccati equation

$$f'(x) + f^2(x) - V(x) + \omega^2 = 0. \tag{2.2}$$

We let  $f$  denote the logarithmic derivative corresponding to an eigenvalue, so that it satisfies the *two* boundary conditions  $f(x) \rightarrow \pm i\omega$  as  $x \rightarrow \pm\infty$ . At a general frequency, however, we can define similar functions  $\phi_{\pm}(\omega, x)$  and their logarithmic derivatives  $f_{\pm}(\omega, x)$  as solutions to (2.1) and (2.2), but with each function satisfying only *one* boundary condition, namely  $f_{\pm}(\omega, x) \rightarrow \pm i\omega$  as  $x \rightarrow \pm\infty$ . At an eigenvalue,  $f_- = f_+ = f$ .

Now let the potential be perturbed

$$V(x) = V_0(x) + \mu V_1(x) \tag{2.3}$$

where  $\mu$  is a formal small parameter. The eigenvalue  $\omega$  and the function  $f$  are both written in powers of  $\mu^\dagger$ :

$$\omega = \omega_0 + \mu\omega_1 + \mu^2\omega_2 + \dots \tag{2.4}$$

$$f = f_0 + \mu f_1 + \mu^2 f_2 + \dots \equiv f_0 + g \tag{2.5}$$

where  $f_0$ , assumed known, satisfies the Riccati equation (2.2) with the potential  $V_0$  and frequency  $\omega_0$ .

Now divide the real line into three regions  $(-\infty, L_-)$ ,  $(L_-, L_+)$  and  $(L_+, \infty)$ . If the original potential and its perturbation both have finite support within the central interval, then the asymptotic regions are trivial, and the simplest examples will be of this type.

First consider the central region, and put (2.4) and (2.5) into (2.2). Upon comparing powers of  $\mu$ , one finds

$$f'_n + 2f_0 f_n + 2\omega_0 \omega_n = V_n \tag{2.6}$$

for  $n = 1, 2, \dots$ , in which  $V_1$  is the perturbing potential in (2.3), and  $V_n$ ,  $n > 1$ , is a shorthand for the following combination of lower-order quantities, to be called the effective  $n$ th-order potential

$$V_n(x) = - \sum_{i=1}^{n-1} [f_i(x) f_{n-i}(x) + \omega_i \omega_{n-i}]. \tag{2.7}$$

Using the integrating factor  $\exp[2 \int^x dy f_0(y)] = \phi_0^2(x)$ , one gets from (2.6)

$$f_n(x) \phi_0^2(x) \Big|_{L_-}^{L_+} = \int_{L_-}^{L_+} dx [V_n(x) - 2\omega_0 \omega_n] \phi_0^2(x). \tag{2.8}$$

We now need to match the central solution to the two asymptotic regions. Assume that the latter have been solved with outgoing wave boundary conditions at spatial infinity, and denote the logarithmic derivatives to be matched as

$$D_{\pm}(\omega) = f_{\pm}(\omega, L_{\pm}). \tag{2.9}$$

Note that  $D_{\pm}$  will contain two types of changes from the unperturbed case. First, at fixed  $\omega$ , the wavefunction when integrated inwards from  $\pm\infty$  will suffer changes because of  $V_1(x)$  in the two asymptotic regions; these are expressed through

$$D_{\pm}(\omega) = D_{\pm 0}(\omega) + \mu D_{\pm 1}(\omega) + \mu^2 D_{\pm 2}(\omega) + \dots \tag{2.10}$$

<sup>†</sup> It is a property of LPT that one need focus only on one state at a time. Therefore, a label for different QNMs will, in general, be suppressed.

Secondly, there will be changes because the value of  $\omega$  itself shifts according to (2.4). In particular, the exact logarithmic derivative is  $f_{\pm}(\omega, L_{\pm}) = D_{\pm}(\omega)$ , whereas the corresponding unperturbed quantity is  $f_0(L_{\pm}) = D_{\pm 0}(\omega_0)$ . Thus

$$g(L_{\pm}) = f_{\pm}(\omega, L_{\pm}) - f_0(L_{\pm}) = D_{\pm}(\omega) - D_{\pm 0}(\omega_0). \quad (2.11)$$

Using (2.4) and (2.10) and developing the right-hand side of (2.11) in powers of  $\mu$ , we can find that  $f_n$  in (2.8) should be matched to

$$f_n(L_{\pm}) = \omega_n D'_{\pm 0}(\omega_0) + \Delta_{\pm n} \quad (2.12)$$

where  $\Delta_{\pm n}$  does not contain  $\omega_n$ ; explicitly but in shorthand

$$\Delta_1 = D_1 \quad \Delta_2 = D_2 + \omega_1 D'_1 + \frac{1}{2} \omega_1^2 D''_0 \quad (2.13)$$

etc. In the above, the subscripts  $\pm$  have been omitted from all quantities, and all  $D_n$  on the right are to be evaluated at  $\omega_0$ . In short, one requires a knowledge of the perturbation in the asymptotic region ( $D_n$ ,  $n > 0$ ), as well as a knowledge of the unperturbed problem slightly away from the original frequency (derivatives of  $D_0$ ).

Putting these into (2.8) and collecting terms involving  $\omega_n$ , one finds the central result

$$\omega_n = \frac{\langle \phi_0 | V_n | \phi_0 \rangle}{2\omega_0 \langle \phi_0 | \phi_0 \rangle} \quad (2.14)$$

in which we have introduced the suggestive notation

$$\langle \phi_0 | V_n | \phi_0 \rangle = \int_{L_-}^{L_+} dx V_n(x) \phi_0^2(x) + [-\Delta_{+n} \phi_0^2(L_+) + \Delta_{-n} \phi_0^2(L_-)] \quad (2.15)$$

$$\langle \phi_0 | \phi_0 \rangle = \int_{L_-}^{L_+} dx \phi_0^2(x) + \frac{1}{2\omega_0} [D'_{+0} \phi_0^2(L_+) - D'_{-0} \phi_0^2(L_-)]. \quad (2.16)$$

This expresses the  $n$ th-order correction in quadrature in terms of lower-order quantities (provided the asymptotic regions have been solved to give  $\Delta_{\pm n}$  and  $D'_{\pm 0}$ ).

The division into three regions is arbitrary, and the whole expression (2.14) must be independent of  $L_{\pm}$ . Moreover, the numerator and denominator must be separately independent of  $L_{\pm}$ , because the numerator depends on the perturbation, whereas the denominator relies only on the unperturbed system; an explicit proof can be constructed by calculating  $\partial \langle \phi_0 | \phi_0 \rangle / \partial L_{\pm}$ , and then using (2.2).

Thus, in both (2.15) and (2.16), we can formally take  $L_{\pm} \rightarrow \pm\infty$  and write

$$\langle \phi_0 | V_n | \phi_0 \rangle = \int_{-\infty}^{\infty} dx V_n(x) \phi_0^2(x) \quad (2.17)$$

$$\langle \phi_0 | \phi_0 \rangle = \int_{-\infty}^{\infty} dx \phi_0^2(x). \quad (2.18)$$

These formal expressions do not converge; equations (2.15) and (2.16) may be regarded as ways of regularizing them. In section 4 we shall discuss various different ways of giving meanings to these formal integrals.

Evidently, the numerator should be regarded as a generalized matrix element, and the denominator should be regarded as a generalized norm. We now develop this interpretation. The corrections to the eigenfunction will be given in section 2.3.

2.2. Generalized norm and matrix element

For the simplest case where  $V_0(x)$  vanishes outside the interval  $(L_-, L_+)$ , the solutions in the two asymptotic regions are exactly  $e^{\pm i\omega x}$ , and  $D_{\pm 0}(\omega) = \pm i\omega$ . The generalized norm (2.16) simplifies to

$$\langle \phi_0 | \phi_0 \rangle = \int_{L_-}^{L_+} dx \phi_0^2(x) + \frac{i}{2\omega_0} [\phi_0^2(L_+) + \phi_0^2(L_-)]. \tag{2.19}$$

In this form applicable to potentials without tails, the generalized norm has been introduced previously both for the wave equation [21, 27], the Schrödinger equation [28] and the Klein–Gordon equation [24], and its properties discussed. It has been shown to be equivalent to another form first given by Zeldovich [29], which did not have the surface terms, but instead required a process of regularization [11–13] which is less convenient for actual evaluation (especially numerical evaluation). The present result, in the more general form (2.16), is however, applicable to potentials with tails, and examples will be given in section 4.

Next we briefly describe the properties of this generalized norm, and argue why it deserves to be so named.

First of all, suppose the system parameters can be tuned so that the leakage of the wavefunction approaches zero (e.g. if  $V_0(x)$  contains a tall barrier on both sides). Then  $L_{\pm}$  can be chosen so that  $\phi_0(L_{\pm}) \approx 0$ ; the expression in (2.16) then contains only the integral. Moreover, when the leakage is zero, the frequency is real, and the wavefunction has a constant phase, which can be chosen to be real; thus  $\phi_0^2 = |\phi_0|^2$ . The expression (2.16) then reduces to the usual (real and positive-definite) norm for a NM. Because of this property, and because it appears in the denominator in (2.14) to scale the wavefunction, it is appropriate to call this quantity the generalized norm.

Nevertheless, it has some unusual properties. (i) It involves  $\phi_0^2$  rather than  $|\phi_0|^2$ , and is, in general, a complex quantity. (ii) It involves a surface term, though the value of the entire expression is independent of the choice of  $L_{\pm}$ . Thus, it is not a genuine norm, and the term is merely a shorthand for ‘a bilinear map that appears in the place of the norm in perturbation formulae such as (2.14)’.

It is hardly surprising that perturbative results are expressed in the form of a matrix element divided by a normalizing factor, as in (2.14), but it would not have been obvious what the normalizing factor should be. The point is that for a QNM, the wavefunction behaves as  $\phi_0(x) \approx e^{i\omega_0 x} \propto e^{\gamma_0 x}$  as  $x \rightarrow \infty$  and  $\gamma_0 = -\text{Im } \omega_0$ , so that an expression such as (2.18) (and even more so for the analogous formula with  $|\phi_0|^2$ ) would be divergent. An important result of the present paper is that we give a precise way of normalizing such QNM wavefunctions.

We have already remarked that the generalized norm is not real, and neither is the (diagonal) matrix element. Far from being a problem, this is necessary, in that the result (2.14) gives the corrections to both the real part and the imaginary part of the frequency. Thus, despite the formal similarity to the analogous problem for NMs, the present formalism in fact contains twice the amount of information.

Some of the properties above, in particular the validity without regularization of the simpler form (2.17), relies on  $V_n$  behaving mildly at infinity. It is therefore appropriate to demonstrate that if the perturbation  $V_1$  has finite support, then so does all the effective potentials  $V_n$  generated via (2.7). Consider for simplicity only the asymptotic interval  $(L_+, \infty)$ . Now the exact logarithmic derivative is  $i\omega$ , whereas the unperturbed analogue is

$i\omega_0$ . This then gives

$$g^2 = -(\omega - \omega_0)^2 = -\left(\sum_{i=1}^{\infty} \mu^i \omega_i\right)^2. \tag{2.20}$$

The  $n$ th-order term in the above expression then ensures that  $V_n$  in (2.7) vanishes exactly in this region.

### 2.3. Wavefunction

To complete the iterative procedure, we also need the eigenfunctions. This can be readily obtained by integrating (2.8) to an arbitrary point, and using (2.11) as the boundary condition, which yields

$$f_n(x) \phi_0^2(x) = [\omega_n D'_{-0}(\omega_0) + \Delta_{-n}] \phi_0^2(L_-) + \int_{L_-}^x dy [V_n(y) - 2\omega_0 \omega_n] \phi_0^2(y). \tag{2.21}$$

One could write an alternate expression using the boundary condition at  $L_+$  and integrating from the right. Consistency is guaranteed if  $\omega_n$  has been correctly evaluated by (2.14).

Thus we have in principle an order-by-order iteration scheme for the QNMs; namely, use (2.14) to obtain  $\omega_1$ , and (2.21) to get  $f_1$ ; this is then put into (2.7) to find  $V_2$ , etc.

## 3. Explicit form of higher-order corrections and an example

### 3.1. Higher-order corrections

The perturbative formulae would be more useful if they could be written explicitly rather than recursively. In general, the  $n$ th-order correction to the frequency must take the form of an integral over  $V(x_1) \cdots V(x_n)$ ; moreover, the perturbing potential can only act if it is ‘sampled’ by the wavefunction  $\phi_0^2(x)$ . It will also turn out to be convenient to remove a constant from  $V_1$ , and we are led to define

$$W(x) = [V_1(x) - 2\omega_0 \omega_1] \phi_0^2(x). \tag{3.1}$$

The constant subtracted renders the integral of  $W$  zero (see equation (2.14)).

Thus we expect to be able to write the  $n$ th-order correction in the form

$$\omega_n = \frac{1}{2\omega_0 \langle \phi_0 | \phi_0 \rangle} \int dx_1 \cdots dx_n S \prod_{j=1}^n W(x_j) \Psi_n(x_1, \dots, x_n) \tag{3.2}$$

where  $S \equiv \theta(x_1 - x_2) \cdots \theta(x_{n-1} - x_n)$  makes use of the symmetry among the coordinates  $x_1, \dots, x_n$  to restrict the integration to one sector ( $\theta$  is the unit step function), and the weight function  $\Psi_n$ , constructed out of  $\phi_0$ , scales as  $(\phi_0^2)^{1-n}$ .

We now try to write out  $\omega_2$  in essentially this form, and evaluate the weight function  $\Psi_2$ . For this purpose we consider the simpler case of a half line  $0 < x < \infty$ , with the potentials satisfying  $V_0(x) = V_1(x) = 0$  for  $x > a$ , and all  $\phi(x = 0) = 0$ . (This may be regarded as the odd-parity sector of a symmetric problem.) Thus, all the surface contributions at  $L_-$  are eliminated, while at the right-hand side we may take  $L_+ = a$  and  $D'_{+0} = i$ . Thus, the generalized norm is

$$\langle \phi_0 | \phi_0 \rangle = \int_0^a dx \phi_0^2(x) + \frac{i}{2\omega_0} \phi_0^2(a). \tag{3.3}$$

By using (2.7) for  $V_2$ , we can write the second-order matrix element as†

$$\langle \phi_0 | V_2 | \phi_0 \rangle = - \int_0^a dx \phi_0^{-2}(x) [f_1(x) \phi_0^2(x)]^2 - \omega_1^2 \int_0^a dx \phi_0^2(x). \quad (3.4)$$

Now from (2.21) we have  $f_1(x) \phi_0^2(x) = \int_0^x dy W(y)$ . Putting this into (3.4), changing the order of integration and also using (3.3) to simplify the second integral in (3.4) then leads to

$$\omega_2 = \frac{1}{2\omega_0 \langle \phi_0 | \phi_0 \rangle} \int dy dz SW(y) W(z) \Psi_2(y, z) - \frac{\omega_1^2}{2\omega_0} + \frac{i\omega_1^2 \phi_0^2(a)}{4\omega_0^2 \langle \phi_0 | \phi_0 \rangle} \quad (3.5)$$

in which the weight function is found to be

$$\Psi_2(y, z) = \Psi_2(y) = -2 \int_y^a dx \phi_0^{-2}(x). \quad (3.6)$$

Thus, except for the last two terms in (3.5), which do not involve an integral, the second-order correction has been cast in the form (3.2), which is the most explicit form possible for an arbitrary perturbation.

### 3.2. Example

We now illustrate these formulae by a very simple example. Let the unperturbed system be defined by a step

$$V_0(x) = V_0 \theta(b - x) \quad b < a. \quad (3.7)$$

The unperturbed eigenfunctions are

$$\phi_0(x) = \begin{cases} A \sin qx & x \leq b \\ A \sin qb e^{i\omega_0(x-b)} & x > b \end{cases} \quad (3.8)$$

where the condition of outgoing waves results in the eigenvalue equation for  $q$ :

$$q \cot qb = i\sqrt{q^2 + V_0} \equiv i\omega_0. \quad (3.9)$$

Here  $\omega_0$  is the unperturbed frequency. There are of course many solutions to (3.9), and we pay attention to any one of these.

The norm is readily evaluated by (3.3) to be

$$\langle \phi_0 | \phi_0 \rangle = \frac{A^2 b}{2} \left( 1 - \frac{\sin 2qb}{2qb} - \frac{\sin^2 qb \tan qb}{qb} \right). \quad (3.10)$$

For any perturbation  $V_1$  with support on  $(0, a)$ , the first-order shift is then

$$\omega_1 = \frac{\int_0^b dx \sin^2 qx V_1(x) + \int_a^b dx \sin^2 qb e^{2i\omega_0(x-b)} V_1(x)}{\omega_0 b (1 - \sin 2qb/2qb - \sin^2 qb \tan qb/qb)} \quad (3.11)$$

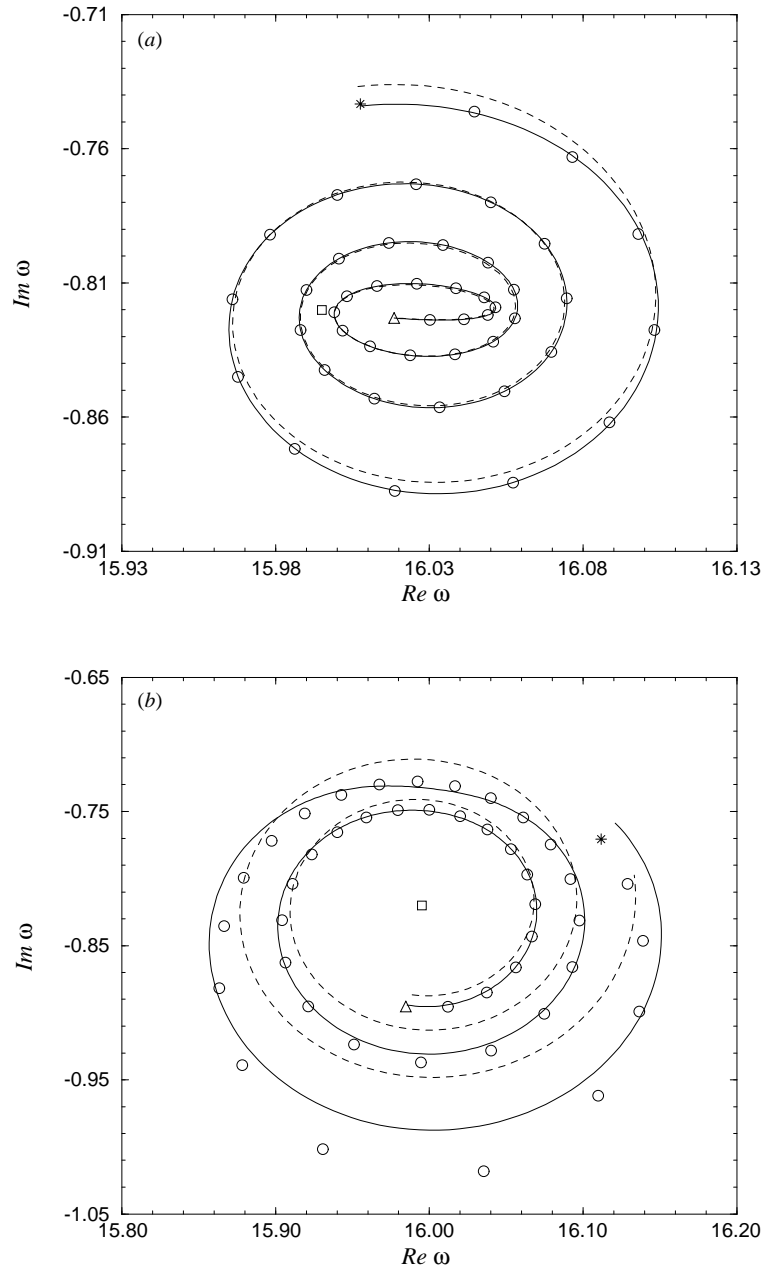
while the second-order correction is given by (3.5), with the weight function being

$$\Psi_2(y) = \begin{cases} (2/A^2)(\cot qb - \cot qy) + C[1 - e^{2i\omega_0(b-a)}] & y < b \\ C(e^{-2i\omega_0 y} - e^{-2i\omega_0 a}) & y \geq b \end{cases} \quad (3.12)$$

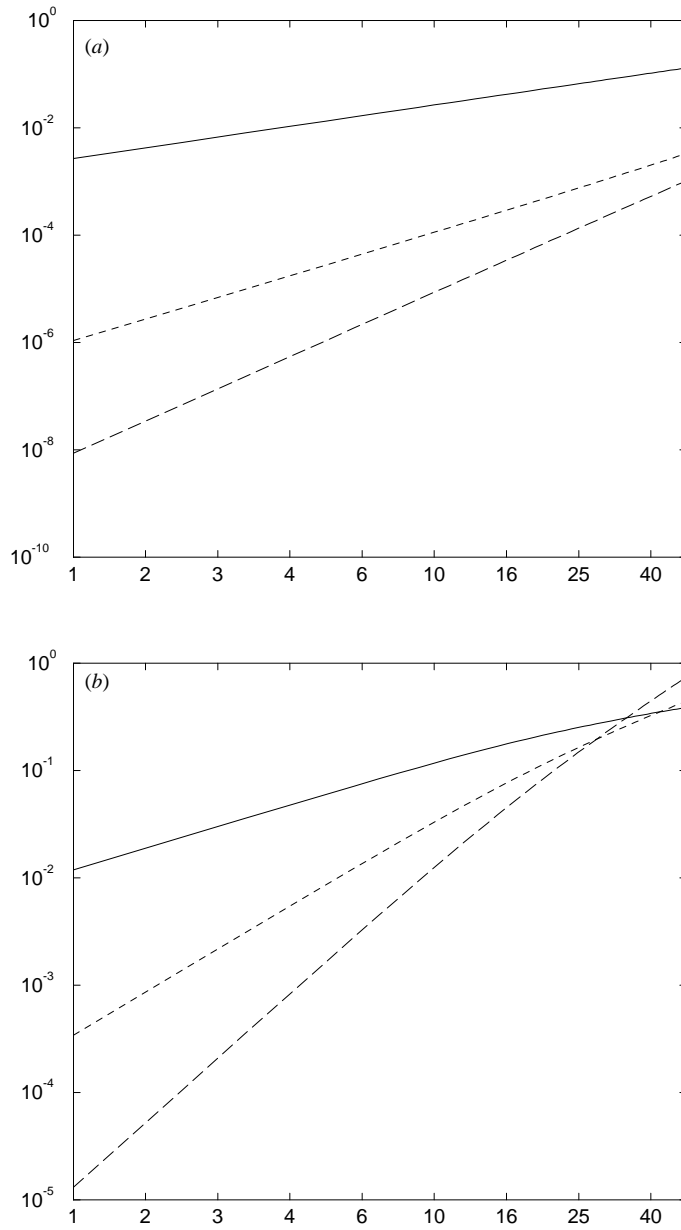
where  $C = i/(\omega_0 A^2 \sin^2 qb)$ .

† For QNMs, the wavefunction does not have any nodes apart from the one at  $x = 0$  imposed by the boundary condition.





**Figure 1.** (a) The trajectory of the lowest eigenvalue in the complex frequency plane as  $x_0$  is changed from 0.05 to 0.95, for fixed  $V_0 = 100$ ,  $b = 1$ ,  $\mu = 10$ ,  $w = 0.1$ . The first- and second-order perturbations are indicated by the broken curve and the full curve, respectively. The triangle and the star mark the positions of the exact eigenvalues for  $x_0 = 0.05$  and  $0.95$ , respectively. The circles show the positions of the exact results for other values of  $x_0$ , and the unperturbed eigenvalue is denoted by the square. (b) Same as (a) but for  $1.05 \leq x_0 \leq 1.50$ . The triangle and the star mark the positions of the exact eigenvalues for  $x_0 = 1.05$  and  $1.50$ , respectively.



**Figure 2.** (a) The magnitude of the remaining error for the unperturbed value (full curve), first-order perturbation (broken curve) and second-order perturbation (long-broken curve) versus  $\mu$ , for fixed  $V_0 = 100$ ,  $b = 1$ ,  $w = 0.1$  and  $x_0 = 0.3$ . (b) Same as (a) but with  $x_0 = 1.4$ .

These formulae then allow the corrections for any perturbation  $V_1$  to be obtained by direct quadrature; more importantly, they exhibit how the perturbation  $V_1$  acts to shift the complex eigenvalues.

To be specific, let the perturbation be a bump of width  $w$  centred at a position  $x_0$ :

$$V_1(x) = \begin{cases} 1 & x_0 - \frac{1}{2}w < x < x_0 + \frac{1}{2}w \\ 0 & \text{otherwise.} \end{cases} \quad (3.13)$$

Figure 1 shows the trajectory of the lowest eigenvalue,  $\omega$ , in the complex frequency plane as  $x_0$  is changed, for fixed  $V_0 = 100$ ,  $b = 1$ ,  $\mu = 10$ ,  $w = 0.1$  and (a)  $V_1$  lying inside the interval  $(0, b)$ , (b)  $V_1$  lying outside the interval  $(0, b)$ ; the exact results (circles), first-order perturbation computed by (3.11) (broken curve) and second-order perturbation computed by (3.5) (full curve) are shown together for comparison. Figure 2 shows the magnitude of the remaining error for the unperturbed eigenvalue (full curve), first-order perturbation (broken curve) and second-order perturbation (long-broken curve) versus  $\mu$ , for fixed  $V_0 = 100$ ,  $b = 1$ ,  $w = 1$  and (a)  $x_0 = 0.3$ , (b)  $x_0 = 1.4$ .

Although this example is extremely simple, it illustrates several interesting features. First, the remaining error of the  $n$ th-order perturbation scales as  $\mu^{n+1}$ , as expected. This is the case even for the perturbation lying outside the interval  $(0, b)$  (figure 2(b)), in which case the spectrum of QNMs is *not* complete, and it is not possible to write the second-order correction as a sum over intermediate QNMs; this result for the second-order correction is testimony to the utility of LPT (as opposed to RSPT).

Comparison between the two cases in figures 2(a) and (b) also reveals that higher-order corrections are more significant when the perturbation acts at a more distant position, where  $|\phi_0|^2$  is large.

The behaviour in figure 1 is even more interesting, showing a spiral structure as the position  $x_0$  is changed. Although the perturbation is real and positive, the shift can have any phase depending on where the perturbation acts—a situation totally different from NMs in conservative systems. This behaviour is most readily understood in first-order perturbation theory, for which the general result (2.14) can be written as

$$\frac{\delta\omega}{\delta V(x)} = H(x) \equiv \frac{\phi_0^2(x)}{\langle \phi_0 | \phi_0 \rangle} \quad (3.14)$$

where  $H(x)$  can be read off from (3.8) and (3.10), which makes the spiral structure easy to understand. The pattern of the shifts can therefore be very rich, and an example of the results for a model astrophysical perturbation of a black hole has been given recently [30].

## 4. Potentials with tails

### 4.1. The Pöschl–Teller potential

When dealing with QNMs, one recurring complexity is the asymptotic behaviour  $\phi_0^2(x) \sim e^{2\gamma_0|x|}$ , making norms and matrix elements divergent. As far as LPT is concerned, this complication occurs at two different levels. If the potential (and its perturbation) vanishes outside a finite domain, then the expression (2.19) suffices to produce a finite expression for the norm, whereas the matrix elements involve integrals only over finite domains. Such simplifications also extend to potentials that vanish at infinity faster than any exponential. However, when the potential (or its perturbation) decays as an exponential or slower (which we shall refer to as a tail), then the evaluation of the norm and the matrix element will require more attention.

In this section we illustrate the solution of these problems with the example of the Pöschl–Teller (P-T) potential [31]

$$V(x) = V_0 \cosh^{-2}(x/b). \quad (4.1)$$

From the point of view of LPT, the P-T potential is interesting because its large  $|x|$  behaviour is exactly exponential:  $V(x) \propto e^{-2x/b}$ . The P-T potential, as one of a few exactly solvable models, has been studied in depth, in part as a proxy for the Regge–Wheeler potential [16, 18, 19] or the Zerilli potential [17], which describes linearized gravitational waves propagating on a Schwarzschild background. These also have exponential tails (as the tortoise coordinate  $x \rightarrow -\infty$ , i.e. towards the event horizon), and consequently their QNMs share certain key properties with those of the P-T potential (e.g. a string of QNMs evenly spaced ‘vertically’ in the complex  $\omega$ -plane,  $-\text{Im} \omega(j) \propto (j + \frac{1}{2})^\dagger$ ). Therefore a better understanding of the exponential tails may be relevant to gravitational waves as well.

The QNM eigenvalues of (4.1) are [31]

$$\omega(j) = \frac{1}{b} \left[ \pm \sqrt{V_0 b^2 - \frac{1}{4}} - i(j + \frac{1}{2}) \right] \tag{4.2}$$

where we have assumed  $4V_0 b^2 > 1$ . The positive (negative) parity sector corresponds to even (odd)  $j$ . For the purpose of illustrating the LPT formalism, we shall focus on the lowest state in each sector, i.e.  $j = 0$  and  $j = 1$ .

Consider perturbations of the width, specifically

$$\frac{1}{b} = 1 + \mu. \tag{4.3}$$

Because the model is exactly solvable for all  $b$ , we immediately obtain the frequencies in powers of  $\mu$ :

$$\begin{aligned} \omega_0 &= \sqrt{V_0 - \frac{1}{4}} - (j + \frac{1}{2})i \equiv \sigma - (j + \frac{1}{2})i \\ \omega_1 &= \frac{1}{4\sigma} - (j + \frac{1}{2})i \end{aligned} \tag{4.4}$$

etc, where for simplicity we only show the one eigenvalue of each pair with  $\text{Re} \omega > 0$ .

We now show how the shift  $\omega_1$  can be obtained from LPT. From (4.1) and (4.3),

$$\begin{aligned} V_0(x) &= V_0 \cosh^{-2} x \\ V_1(x) &= -2V_0 x \sinh x \cosh^{-3} x. \end{aligned} \tag{4.5}$$

We show three different ways of handling the divergent integrals (2.17) and (2.18). The first two methods are specific to the P-T potential (or other potentials amenable to analytic treatment), but these lead to the third method, which is numerical and can be applied to any potential with exponential tails. The last method will be the one of general interest.

#### 4.2. Analytic continuation

Consider for example the  $j = 0$  state. The unperturbed eigenfunction is<sup>‡</sup>

$$\phi_0(x) = (\cosh x)^{i\omega} \quad j = 0 \tag{4.6}$$

where it is understood that  $\omega$  is to be evaluated at the unperturbed value  $\omega = \omega_0 = \sigma - i/2$ . Take the formal expression (2.18) and define, for any  $\omega$  for which the integral converges,

$$N(\omega) = \int_{-\infty}^{\infty} dx \phi_0^2(x) \tag{4.7}$$

<sup>†</sup> The mode index will be indicated in  $(\ )$ , but will be suppressed where no confusion arises.

<sup>‡</sup> This expression applies only for  $\omega$  equal to the eigenvalue; otherwise there is another term with an incoming wave, whose coefficient vanishes at the eigenvalue. The omission of this term does not affect the argument based on analytic continuation.

where  $\phi_0$  is given by (4.6). The integral  $N(\omega)$  is well defined for  $\text{Im } \omega > 0$ , in which domain it is evaluated in terms of the beta function  $B$  to be

$$N(\omega) = B\left(\frac{1}{2}, -i\omega\right). \quad (4.8)$$

By analytic continuation, this applies to  $\text{Im } \omega < 0$  as well, and the norm of the  $j = 0$  state is then  $B\left(\frac{1}{2}, -\frac{1}{2} - i\sigma\right)$ . Likewise, the wavefunction for the  $j = 1$  state is

$$\phi_0(x) = \tanh x (\cosh x)^{i\omega} \quad j = 1. \quad (4.9)$$

The same analytic continuation gives the norm as  $B\left(\frac{3}{2}, -\frac{3}{2} - i\sigma\right)$ .

The matrix element  $\langle \phi_0 | V_1 | \phi_0 \rangle$  for the  $j = 0$  state is

$$-2V_0 \int_{-\infty}^{\infty} dx x \sinh x (\cosh x)^{2i\omega-3} = -\frac{\sqrt{\pi} V_0 \Gamma(1-i\omega)}{(1-i\omega)\Gamma\left(\frac{3}{2}-i\omega\right)} \quad (4.10)$$

which is convergent even at  $\omega = \omega_0 = \sigma - i/2$ , and readily evaluated to be

$$\langle \phi_0 | V_1 | \phi_0 \rangle = -\frac{\sqrt{\pi} V_0 \Gamma\left(\frac{1}{2}-i\sigma\right)}{\left(\frac{1}{2}-i\sigma\right)\Gamma(1-i\sigma)}.$$

Thus the first-order shift is obtained from (4.8) and (4.10) to be

$$\omega_1 = \frac{1}{4\sigma} - \frac{1}{2}i$$

in agreement with (4.4).

Similarly, for the  $j = 1$  state, the matrix element is

$$-2V_0 \int_{-\infty}^{\infty} dx x \sinh^3 x (\cosh x)^{2i\omega-5} = \frac{V_0}{i\omega-2} \left[ B\left(\frac{3}{2}, 1-i\omega\right) + \frac{\pi \Gamma(1-i\omega)}{(1-i\omega)\Gamma\left(\frac{3}{2}-i\omega\right)} \right] \quad (4.11)$$

where the integral is evaluated for  $\text{Im } \omega > 0$ . Analytic continuation to the eigenvalue is required, and gives

$$\langle \phi_0 | V_1 | \phi_0 \rangle = \frac{V_0}{i\sigma - \frac{1}{2}} \left[ B\left(\frac{3}{2}, -\frac{1}{2} - i\sigma\right) - \frac{\sqrt{\pi} \Gamma\left(-\frac{1}{2} - i\sigma\right)}{\left(\frac{1}{2} + i\sigma\right)\Gamma(-i\sigma)} \right].$$

The first-order shift of the  $j = 1$  state is again in agreement with (4.4).

Analytic continuation, though convenient, applies only when the integrals can be evaluated exactly. We therefore present other methods, including numerical evaluation of the integrals.

### 4.3. Regularization

The second method does not make use of the formal expression (2.18), but utilizes the original expression (2.16) with the regulating parameters  $-L_- = L_+ = L$ . The integral involved is, for the  $j = 0$  state,

$$\begin{aligned} \int_{-L}^L dx \phi_0^2(x) &= \int_0^{\tanh^2 L} dz z^{-1/2} (1-z)^{-i\omega_0-1} \\ &= 2F\left(\frac{1}{2}, 1+i\omega_0; \frac{3}{2}; \tanh^2 L\right) \tanh L \\ &= B\left(\frac{1}{2}, -i\omega_0\right) - \frac{i\phi_0^2(L)}{\omega_0} F\left(1, \frac{1}{2} - i\omega_0; 1 - i\omega_0; \cosh^{-2} L\right) \tanh L \end{aligned} \quad (4.12)$$

where  $F(a, b; c; x)$  is the hypergeometric function, and the last step follows from its transformation properties. This then gives

$$\langle \phi_0 | \phi_0 \rangle = B\left(\frac{1}{2}, -i\omega_0\right) - \frac{\phi_0^2(L)}{\omega_0} K(L) \quad (4.13)$$

where

$$K(L) = D'_+ - iF(1, \frac{1}{2} - i\omega_0; 1 - i\omega_0; \cosh^{-2} L) \tanh L. \tag{4.14}$$

Now it is easily shown, using even the crudest approximation  $D_+ = i\omega_0$ , that  $K(L) = O(e^{-2L})$ , whereas  $\phi_0^2(L) = O(e^L)$ . Thus the second term in (4.13) (which is guaranteed to be independent of  $L$ ) is shown to be zero when evaluated at  $L \rightarrow \infty$ . The norm of the  $j = 1$  state can also be recovered in this manner, though one needs a better approximation for  $D_+$  in this case; a way to obtain these better approximations is given below.

The matrix elements are likewise regulated, provided we know the logarithmic derivatives  $D_\pm(\omega)$  when the wavefunction is integrated from  $x \rightarrow \pm\infty$ . These are not available for a general potential. Another way to regulate the matrix elements is to integrate along a contour in complex  $x$  plane, e.g. along the path  $x = ue^{i\theta}$ ,  $u$  real and  $\theta$  fixed. The matrix element becomes

$$\langle \phi_0 | V_0 | \phi_0 \rangle = \int_{-\infty}^{\infty} du e^{i\theta} \phi_0^2(ue^{i\theta}) V_0(ue^{i\theta}). \tag{4.15}$$

It is easily seen that  $\phi_0^2(ue^{i\theta})$  decays exponentially for sufficiently large  $\theta$  and hence the integral along the rotated contour converges [13]. As an example, we compute the matrix element numerically for  $j = 1$  state using (4.15) with  $\theta = 60^\circ$ . The result agrees with the analytic value given by (4.11). In the case of P-T potential, the logarithmic derivatives  $D_\pm(\omega)$  are available analytically. Hence the matrix element can also be evaluated by (2.15), but since this result is highly special, we shall not exhibit it here. Instead, we go on to a numerical scheme applicable to all potentials with exponential tails.

#### 4.4. Numerical evaluation and Born series

For any system, provided the unperturbed wavefunction  $\phi_0(x)$  is known, the shifts are in principle given by (2.15) and (2.16). These involve (i) finite integrals from  $L_- = -L$  to  $L_+ = L$ , which can be handled numerically in the usual way and (ii) surface contributions involving  $D_\pm$ . The latter contain all the information from the tails of the potential. The lowest approximation  $D_\pm(\omega) \approx \pm i\omega$  is, in general, not accurate enough, because it multiplies  $\phi_0^2(L) \sim e^{2\gamma_0 L}$ .

A general, yet simple way to obtain a better approximation for  $|x| > L$  is to use the Born approximation. Here  $V$  will stand for *any* potential; by applying the method sketched below, we can find the logarithmic derivative of either  $V_0$  or  $V_0 + V_1$ , and hence obtain the quantities  $D_\pm$ . If  $V(x) \propto e^{-\alpha x}$  and the Born approximation is carried to  $m$ th order, then the remaining error would go as  $V(x)^{m+1} \propto e^{-(m+1)\alpha x}$ , which will be sufficiently accurate for dealing with any unperturbed state with  $\gamma_0 < (m + 1)\alpha$ .

The Born approximation is particularly easy to implement for a potential that goes as an exponential. For simplicity we deal with the tail at  $x = L$  only. Let

$$V(x) = V_0 \sum_k c_k e^{-\alpha_k x} \tag{4.16}$$

where by convention  $\alpha_1 < \alpha_2 < \dots$ . The P-T potential is of this form, where  $\alpha_k = 2k/b$ , and  $c_k = (-1)^{k+1} 4k$ .

By iterating (2.2) in powers of  $V$ , one finds

$$\begin{aligned} f_0(x) &= i\omega \\ f_1(x) &= V_0 \sum_k c_k (\alpha_k - 2i\omega)^{-1} e^{-\alpha_k x} \\ f_2(x) &= V_0 \sum_{k,k'} c_k c_{k'} (\alpha_k - 2i\omega)^{-1} (\alpha_{k'} - 2i\omega)^{-1} (\alpha_k + \alpha_{k'} - 2i\omega)^{-1} e^{-(\alpha_k + \alpha_{k'})x}. \end{aligned} \tag{4.17}$$

etc. (Here the subscripts on  $f$  denote the order of the Born approximation, not LPT.) Higher-order terms can be generated readily by algebraic software. All the sums can be terminated at some  $k_{max}$  if only accuracy up to  $O(e^{-\beta x})$  is required for some finite  $\beta$ .

The poles in (4.17) exist only in the Born approximation, but not in the exact solution<sup>†</sup>. Nevertheless, equation (4.17) makes it clear that even exponentially small potentials can have a significant effect when  $-\text{Im}\omega$  is large, and this is the reason behind the string of QNMs  $-\text{Im}4M\omega \approx (k + \frac{1}{2})$  for a Schwarzschild black hole of mass  $M$  [32].

We have implemented this scheme for the P-T potential, indeed for any potential that can be expressed in the form (4.16) with  $\alpha_k = k\alpha$ . Again for simplicity we deal with the situation only on one side, say for the tail as  $x \rightarrow +\infty$ . For this particular form of  $\alpha_k$  spaced evenly in  $k$ , the Klein–Gordon equation can be solved easily by substituting

$$\phi(x) = e^{i\omega x} \sum_{k=0}^{\infty} d_k e^{-k\alpha x} \quad (4.18)$$

into (2.1). One has

$$\begin{aligned} d_0 &= 1 \\ d_k &= \frac{V_0}{\alpha k(\alpha k - 2i\omega)} \sum_{m=0}^{k-1} d_m c_{k-m} \quad k \geq 1 \end{aligned} \quad (4.19)$$

and

$$f(x) = \left[ \sum_{k=0}^{\infty} (i\omega - \alpha k) d_k e^{-\alpha k x} \right] \left( \sum_{k=0}^{\infty} d_k e^{-\alpha k x} \right)^{-1}. \quad (4.20)$$

In this example, we have summed four terms with  $\alpha = 2/b = 2$ , and consequently the remaining error in the logarithmic derivative calculated is  $O(e^{-10x})$ . Incidentally, this method, when evaluated at  $x = L$ , gives accurate expression for  $D_+(\omega)$  as needed in section 4.3.

We have used this method to evaluate both the matrix element (2.15) and the norm (2.16), taking  $L_+ = L = 5$ . (In this example, only the positive half line is needed due to the symmetry of the potential.) To be precise, the integral over the finite domain  $(0, L)$  is evaluated numerically, while the surface term is evaluated by the Born series (4.20). The result for  $\omega_1$  agrees accurately with the result obtained from the two methods sketched earlier.

However, there is still a numerical problem. Take the norm in the  $j = 1$  state as an example. Numerical evaluation gives for the two terms in (2.15)

$$\begin{aligned} \text{integral} &= 187\,374.578 + 143\,350.152i \\ \text{surface term} &= -187\,374.961 - 143\,350.431i \end{aligned}$$

so that there is a loss of six significant digits when the two terms are combined. The cause of the problem, as before, is the exponential growth of the wavefunction  $\phi_0(x) \approx Ae^{i\omega_0 x}$ , so that the asymptotic  $L$  dependence of the two terms are respectively  $\pm(A^2/2i\omega_0)e^{2i\omega_0 L}$ , where  $A = 1/2^{i\omega_0}$  for the wavefunction  $\phi_0(x)$  normalized as in (4.9). A related difficulty is that the integrand is large and oscillating, which limits the accuracy of evaluating the integral. However, these difficulties are readily remedied if we subtract  $A^2e^{2i\omega_0 x}$  from the

<sup>†</sup> In deriving the Born approximation, one has in effect first taken  $V_0 \rightarrow 0$ , then secondly considered say  $\alpha_k - 2i\omega \rightarrow 0$  in the resultant expression (4.17). This order of the limits implies that the result is only valid for  $V_0e^{-\alpha_k x} \ll |\alpha_k - 2i\omega|$ . On the other hand, the exact solution at the position in question would refer to taking the limit  $\alpha_k - 2i\omega \rightarrow 0$  while keeping  $V_0$  finite. In this case, there would be no pole.

integrand, and add the corresponding term  $(A^2/2i\omega_0)(e^{2i\omega_0 L} - 1)$  to the surface term. Then in this example one finds

$$\text{modified integral} = -22.937\,0946 + 29.121\,5523i$$

$$\text{modified surface term} = 22.554\,1253 - 29.401\,0213i$$

and there is only a loss of two significant digits when the two terms are combined. This technique can be further refined by removing subasymptotic terms as well. This method does not rely on any property of the P-T potential other than the exponential tails.

The numerical difficulty associated with the exponential growth of the QNM wavefunction is exactly the same as the difficulty in the ‘shooting’ algorithm discussed in section 1.2. This same difficulty, in different guises, always besets numerical solutions of QNM problems. Here we have developed an effective method within the realm of perturbation theory—but otherwise applicable to *any* system with exponential tails—to tame the problem. The class of problems with exponential tails is sufficiently wide for this method to be of interest, especially since there is a dearth of other effective methods. With this numerical technique to handle exponential tails, LPT is completely formulated for potentials either without tails, or with exponential tails.

## 5. Conclusion

In this paper we have formulated LPT for QNMs. For systems without tails, the formalism is no more complicated than for NMs. In fact, there are several advantages: the absence of nodes allows simple application to *all* states, not just the ground state, and the possibility that QNMs may not be complete makes alternative methods (e.g. generalization of RSPT) less useful. The explicit form of the first-order shift is given, as well as the most general form of the second-order shift for an arbitrary perturbation. When there is a tail that can be expressed as a sum of exponentials, a method is developed, based on the Born series, that reduces the calculation to the evaluation of integrals, the exponentially large nature of which can be handled by subtracting off the leading asymptotic terms. While this is somewhat involved, it is to be stressed that for this case *no* other methods apply in general, not even brute-force numerical integration, on account of the need to extract an exponentially small ‘wrong’ solution. Thus the technique is likely to be useful. Indeed, this technique has already been employed to deal with model perturbation of a black hole [30].

Finally, the generalized norm plays a central role, and in fact has a significance beyond perturbation theory. It emerges naturally in the derivation, where the integrating factor in (2.8) is  $\phi_0^2$  and not  $|\phi_0|^2$ . However, it is possible to express this same idea in another way, which is possibly more natural and familiar [33]. The idea is to write these open systems in terms of a non-Hermitian Hamiltonian [25, 26] and adopt a bi-orthogonal basis [34] which includes a set of left-eigenfunctions  $\bar{\phi}$  dual to the right-eigenfunctions  $\phi$ . Then our generalized norm  $\langle\phi|\phi\rangle$  is exactly the same as  $(\bar{\phi}, \phi)$ , where the latter is the *conventional* inner product which is conjugate linear in the bra and linear in the ket. This development will be reported elsewhere [35].

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